CONDENSED LESSON

2.1 Proportions

In this lesson you will

- learn several ways for writing a **ratio**
- learn methods for **solving proportions**
- solve problems by writing and solving proportions

The statement "Jackie scored 24 of the team's 64 points" compares two numbers. The **ratio** of the points Jackie scored to the points the team scored is 24 to 64. You can write the ratio as 24 : 64 or as a fraction or decimal. The fraction bar means division, so these expressions are equivalent.

$\frac{24}{64}$ $\qquad 24 \div 64 \qquad 0.375 \qquad \frac{3}{8}$

Read Example A and the text that follows it starting on page 96 of your book. Make sure you understand the difference between a **terminating decimal** and a **repeating decimal.**

A **proportion** is an equation stating that two ratios are equal. Here are some true proportions using the integers 3, 5, 9, and 15.

$\frac{9}{15} = \frac{3}{5} \qquad \frac{15}{9} = \frac{5}{3} \qquad \frac{5}{15} = \frac{3}{9} \qquad \frac{15}{5} = \frac{9}{3}$

You can check that the proportions are true by finding the decimal equivalent of each side. The proportion $\frac{3}{15} = \frac{5}{9}$ is not true; 0.2 is not equal to $0.\overline{5}$.

In algebra, a **variable** stands for an unknown number or numbers. In the proportion $\frac{R}{16} = \frac{1}{4}$, you can replace the variable R with any number, but only one number, 4, will make the proportion true.

Investigation: Multiply and Conquer

Steps 1–4 When you multiply both sides of an equation by the *same number,* the two sides remain equal to each other. You can use this idea to solve proportions with a variable in one of the numerators. For example, you can solve $\frac{M}{19} = \frac{56}{133}$ by multiplying both sides by 19.

$$\frac{M}{19} = \frac{56}{133}$$

$$19 \cdot \frac{M}{19} = \frac{56}{133} \cdot 19 \qquad \text{Multiply both sides by 19.}$$

$$M = \frac{56}{133} \cdot 19 \qquad \frac{19}{19} \text{ is equivalent to 1.}$$

$$M = 8 \qquad \text{Multiply and divide.}$$

You can check that the solution is correct by replacing M with 8 and making sure the resulting proportion, $\frac{8}{19} = \frac{56}{133}$, is true.

(continued)

Here is the solution to part a in Step 2. Try solving parts b–d on your own.

$$\frac{p}{12} = \frac{132}{176}$$

$$12 \cdot \frac{p}{12} = \frac{132}{176} \cdot 12 \qquad \text{Multiply both sides by 12.}$$

$$p = \frac{132}{176} \cdot 12 \qquad \tfrac{12}{12} \text{ is equivalent to 1.}$$

$$p = 9 \qquad \text{Multiply and divide.}$$

Steps 5–7 In Step 5, the ratios in the proportions from Step 2 have been *inverted*. These new proportions have the same solutions as the original proportions. For example, 9 is a solution to both $\frac{p}{12} = \frac{132}{176}$ and $\frac{12}{p} = \frac{176}{132}$. (Check that this is true.) You can use this idea to solve proportions with the variable in a denominator. For example, to solve $\frac{20}{135} = \frac{12}{k}$, just invert the ratios to get $\frac{135}{20} = \frac{k}{12}$, and multiply both sides by 12.

Now, read the question and sample solutions in Step 7 and make sure you understand them.

When a problem involves a ratio or percent, you can sometimes solve it by setting up and solving a proportion. Examples B and C in your book present some sample problems. Here is another example.

EXAMPLE Raj answered 75% of the questions on the algebra midterm correctly. If he got 27 correct answers, how many questions were on the test?

▶ ***Solution*** Let q represent the number of questions on the test. Use the fact that 75% is 75 out of 100 to help you write a proportion. The ratio 27 out of q equals 75 out of 100.

$$\frac{27}{q} = \frac{75}{100} \qquad \text{Write the proportion.}$$

$$\frac{q}{27} = \frac{100}{75} \qquad \text{Invert both sides.}$$

$$q = \frac{100}{75} \cdot 27 \qquad \text{Multiply by 27 to undo the division.}$$

$$q = 36 \qquad \text{Multiply and divide.}$$

There were 36 questions on the test.

CONDENSED LESSON

2.2 Capture-Recapture

In this lesson you will

- simulate the **capture-recapture method** for estimating animal populations
- write and **solve proportions**
- solve three types of **percent problems:** finding an unknown percent, finding an unknown total, and finding an unknown part

Wildlife biologists use a method called "capture-recapture" to estimate animal populations. This method involves tagging some animals and then releasing them to mingle with the larger population. Later, a sample is taken. Using the ratio of tagged animals in the sample to total animals in the sample, biologists can estimate the animal population.

Investigation: Fish in the Lake

In this investigation a bag of white beans represents a population of fish in a lake. To simulate the capture-recapture method, you reach into the "lake" and remove a handful of "fish." You count the fish in the sample. Instead of putting them back, you replace these fish (white beans) with an equal number of "tagged fish" (red beans).

You then allow the fish to mingle (seal the bag and shake it to mix the beans) and then remove another sample. You count all the fish in the sample and the tagged fish in the sample before you return the fish to the lake. By taking several more samples, you can get a good idea of the ratio of tagged fish to total fish in the lake.

One group of students tagged and returned 84 fish. They then took five samples. Here are their results.

Sample number	Number of tagged fish	Total number of fish	Ratio of tagged fish to total fish
1	8	48	$\frac{8}{48} \approx 0.17$
2	24	102	$\frac{24}{102} \approx 0.24$
3	16	86	$\frac{16}{86} \approx 0.19$
4	17	67	$\frac{17}{67} \approx 0.25$
5	16	75	$\frac{16}{75} \approx 0.21$

To estimate the population of fish in this group's lake, you need to choose one ratio to represent all the samples. You might calculate the median or the mean or use some other method of choosing a representative ratio. For this example, we will use the median of the ratios, which is $\frac{16}{75}$.

(continued)

If the fish were mixed well, the fraction of the tagged fish in a sample should be close to the fraction of the tagged fish in the entire population. In other words, the following should be true:

$$\frac{\textit{tagged fish in sample}}{\textit{total fish in sample}} \approx \frac{\textit{tagged fish in population}}{\textit{total fish in population}}$$

In this case, there were 16 tagged fish in the sample, 75 total fish in the sample, and 84 tagged fish in the population. So you can estimate the fish population, *f*, by solving this proportion:

$$\frac{16}{75} = \frac{84}{f}$$

To solve the proportion, invert the ratios and multiply by 84 to undo the division.

$$\frac{75}{16} = \frac{f}{84}$$ Invert both ratios.

$$84 \cdot \frac{75}{16} = f$$ Multiply by 84 to undo the division.

$$393.75 = f$$ Multiply and divide.

So there are about 400 fish in the lake (that is, about 400 beans in the bag).

You can describe the results of capture-recapture situations using percents. The examples in your book show three different kinds of percent problems: finding an unknown percent, finding an unknown total, and finding an unknown part. Be sure to read each example and make sure you understand it. Here is one more example.

EXAMPLE In a lake with 350 tagged fish, the recapture results show that 16% of the fish are tagged. About how many fish are in the lake?

▶ ***Solution*** In this case, the variable is the total number of fish in the lake, *f*. Because 16% of the fish are tagged, there are 16 tagged fish out of every 100 fish. You can write this as the ratio $\frac{16}{100}$. So the ratio of the total number of tagged fish, 350, to the total number of fish in the lake, *f*, is about $\frac{16}{100}$.

$$\frac{16}{100} = \frac{350}{f}$$ Write the proportion.

$$\frac{100}{16} = \frac{f}{350}$$ Invert both ratios.

$$350 \cdot \frac{100}{16} = f$$ Multiply by 350 to undo the division.

$$2187.5 = f$$ Multiply and divide.

There are about 2200 fish in the lake.

CONDENSED LESSON

2.3 Proportions and Measurement Systems

In this lesson you will

- find a **conversion factor** to change measurements from centimeters to inches
- use **dimensional analysis** to do conversions involving several steps

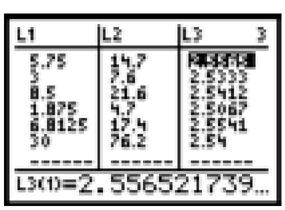
If you travel outside the United States, it is helpful to be familiar with the Système Internationale, or SI, known in the United States as the metric system. To change from one system of measurement to another, you can use ratios called **conversion factors.**

Investigation: Converting Centimeters to Inches

To find a ratio you can use to convert centimeters to inches and inches to centimeters, first carefully measure the length or width of several different objects in both units. Here are some sample data. You may want to collect your own data or just add a few measurements to this table.

Object	Measurement in inches	Measurement in centimeters
pen	$5\frac{3}{4} = 5.75$	14.7
calculator	3.0	7.6
paper	$8\frac{1}{2} = 8.5$	21.6
paper clip	$1\frac{7}{8} = 1.875$	4.7
pencil	$6\frac{13}{16} = 6.81$	17.4
desk	30.0	76.2

Enter the measurements in inches into list L_1 of your calculator and the measurements in centimeters into list L_2. Enter the ratio of centimeters to inches, $\frac{L_2}{L_1}$, into list L_3, and let your calculator fill in the ratio values. (See **Calculator Note 1K.**) Here is the table for the data above.



To find a single value to represent the ratio of centimeters to inches, you can use the median or mean of the ratios in list L_3. In this case, both the mean and median are about 2.54. So the ratio of centimeters to inches is $\frac{2.54}{1}$ or 2.54. This number is the conversion factor between inches and centimeters. It means that 1 inch is equal to about 2.54 centimeters.

(continued)

Using this ratio, you can write and solve a proportion to convert a centimeter measurement to an inch measurement or vice versa. When you set up a proportion, make sure both sides show centimeters to inches or both sides show inches to centimeters.

Here's how to convert 215 centimeters to inches.

$$\frac{2.54}{1} = \frac{215}{x}$$

$$\frac{1}{2.54} = \frac{x}{215}$$

$$215 \cdot \frac{1}{2.54} = x$$

$$84.6 \approx x$$

215 centimeters is about 84.6 inches.

Here's how to convert 80 inches to centimeters.

$$\frac{2.54}{1} = \frac{x}{80}$$

$$80 \cdot \frac{2.54}{1} = x$$

$$203.2 = x$$

80 inches is about 203.2 centimeters.

Some conversions require several steps. Example B in your book shows how to use a strategy called **dimensional analysis** to convert a measurement from feet per second to miles per hour. Here is another example that uses dimensional analysis.

EXAMPLE

A car traveled 500 kilometers on 45 liters of gas. Using the facts that 1 mile equals 1.61 kilometers and 1 gallon equals 3.79 liters, express the car's gas consumption in miles per gallon.

▶ ***Solution***

You can use the given information to express the car's gas consumption as the ratio $\frac{500 \text{ kilometers}}{45 \text{ liters}}$. Using the other facts in the problem, you can create fractions with a value of 1, for example, $\frac{1 \text{ mile}}{1.61 \text{ kilometers}}$. By multiplying the original ratio by such fractions, you can convert the gas-consumption ratio to miles per gallon.

$$\frac{500 \cancel{\text{kilometers}}}{45 \cancel{\text{liters}}} \cdot \frac{3.79 \cancel{\text{liters}}}{1 \text{ gallon}} \cdot \frac{1 \text{ mile}}{1.61 \cancel{\text{kilometers}}} = \frac{1895 \text{ miles}}{72.45 \text{ gallons}}$$

$$\approx \frac{26 \text{ miles}}{1 \text{ gallon}} \text{ or 26 miles per gallon}$$

Notice that the particular fractions equivalent to 1 were chosen such that when the units cancel, the result has miles in the numerator and gallons in the denominator.

26 miles per gallon is a **rate** because it has a denominator of 1. (26 miles per gallon can be written $\frac{26 \text{ miles}}{1 \text{ gallon}}$.) Other examples of rates are a speed of 65 miles per hour, an allowance of 5 dollars per week, or a cost of 75 cents per pound.

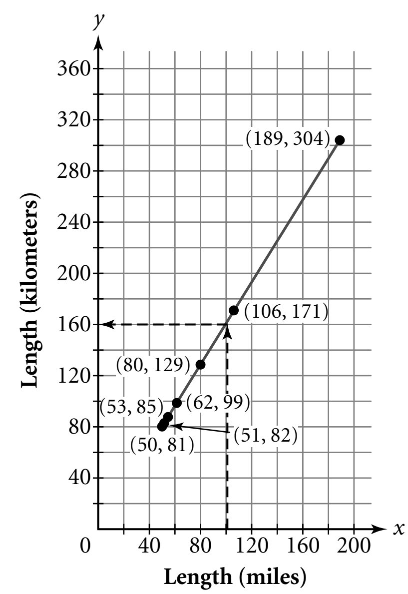
CONDENSED
LESSON
2.4 Direct Variation

In this lesson you will

- represent relationships using graphs, tables, and equations
- use graphs, tables, and equations to find missing data values
- learn about the relationship among **rates, ratios,** and **conversion factors**
- learn about **directly proportional** relationships and **direct variations**

Investigation: Ship Canals

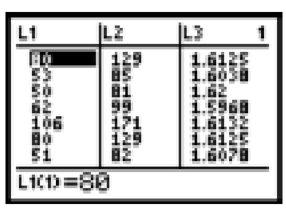
The table on page 114 of your book shows the lengths, in both miles and kilometers, of the world's longest ship canals. Two values are missing from the table. In this investigation you'll learn several ways to find the missing values.

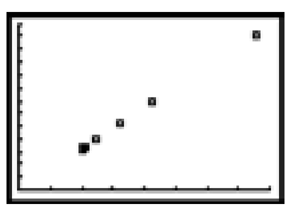


Steps 1–2 This graph shows the data for the first eight canals listed. Connecting the points helps you better see the straight-line pattern.

You can use the graph to estimate the length in kilometers of the Suez Canal. Because the length in miles is 101, start at 101 on the x-axis and move straight up until you reach the line. Then, move straight across to the y-axis, and read off the value. The length is about 160 kilometers. Use a similar method to estimate the length of the Trollhätte Canal in miles.

Steps 3–5 Follow Steps 3 and 4 in your book. When you finish, the List and Graph windows of your calculator should look like this.





[0, 200, 25, 0, 325, 25]

List L_3 represents the ratio of kilometers to miles. Each value in this list rounds to 1.6, so there are about 1.6 kilometers in every mile. You can use this conversion factor to find the missing values in the table.

To find the length of the Suez Canal in kilometers, solve this proportion.

$$\frac{1.6 \text{ kilometers}}{1 \text{ mile}} = \frac{t \text{ kilometers}}{101 \text{ miles}}$$

To find the length of the Trollhätte Canal in miles, solve this proportion.

$$\frac{1.6 \text{ kilometers}}{1 \text{ mile}} = \frac{87 \text{ kilometers}}{t \text{ miles}}$$

(continued)

Steps 6–11 There are 1.6 kilometers per mile. So, to change x miles to y kilometers, multiply x by 1.6. You can write this as the equation $y = 1.6x$.

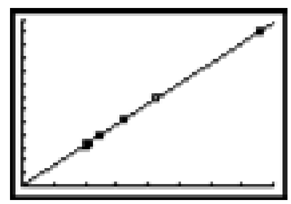
To find the length of the Suez Canal in kilometers, substitute its length in miles for x and solve for y.

$y = 1.6x$	Write the equation.
$y = 1.6 \cdot 101 = 161.6$	Substitute 101 for x and multiply.

To find the length of the Trollhätte Canal in miles, substitute its length in kilometers for y and solve for x.

$y = 1.6x$	Write the equation.
$87 = 1.6x$	Substitute 87 for y.
$\frac{87}{1.6} = x$	To isolate x, divide by 1.6 to undo the multiplication.
$54.375 = x$	Divide.

Now, graph the equation $y = 1.6x$ on your calculator, in the same window as the plotted points.

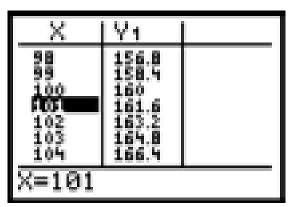


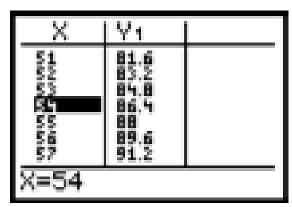
[0, 200, 25, 0, 325, 25]

The line goes through the origin because 0 miles = 0 kilometers. Estimate the length of the Suez Canal in miles by tracing the graph and finding the y-value when the x-value is about 101. (See **Calculator Note 1J.**) Then, estimate the length of the Trollhätte Canal in miles by finding the x-value when the y-value is about 87. Your estimates should be close to those you calculated or found using your hand-drawn graph.

Look at your calculator's table display. (See **Calculator Note 2A.**) To find the length of the Suez Canal in miles, scroll down to the x-value 101. The corresponding y-value is 161.6.

To find the length of the Trollhätte Canal in miles, scroll up to show y-values near 87. Using x-increments of 1, the closest y-value to 87 is 86.4. This gives a miles estimate of 54. To find a closer estimate, you can adjust the table setting to show smaller increments.





You have used several methods for finding the missing values. Which methods do you prefer?

The relationship between kilometers and miles is an example of a type of relationship called a **direct variation.** In a direct variation, the ratio of two variables is constant. Read the text following the investigation carefully. Make sure you understand the terms **directly proportional** and **constant of variation.** Then, read and follow along with the example.

CONDENSED
LESSON
2.5 Inverse Variation

In this lesson you will

- study relationships in which two variables are **inversely proportional**
- write equations for **inverse variations**
- use inverse variation equations to solve problems

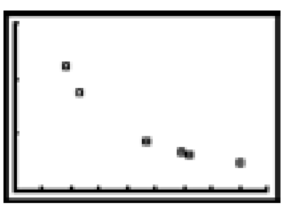
Investigation: Speed versus Time

In this investigation you will compare the time it takes to walk 2.0 meters to the average speed at which the person walks. If you have a partner and a motion sensor, follow Steps 1–5 to collect data.

Steps 1–5 Follow along with Steps 1–5 in your book. The data collected should look something like this

Walk number	Total time (s)	Average speed (m/s)
1	6.2	0.322
2	8	0.250
3	2.3	0.870
4	1.8	1.111
5	4.7	0.426
6	5.9	0.339

Steps 6–8 Here is a graph of the data.



[0, 9, 1, 0, 1.5, 0.5]

Now try to find an equation in the form $y = \frac{a}{x}$ that is a good model for the data.

If you multiply the total time by the average speed values given in the data table, you'll find that the product is always close to 2. What does this value have to do with the experiment? It's the distance walked—2.0 meters. So *time* · *speed* = *distance.* If you rearrange this equation, you get $speed = \frac{2}{time}$, or $y = \frac{2}{x}$. Graph this equation with the data to show that it is a good fit.

You can also use the fact that the product of time and speed is constant to write an equation such as this.

first walk total time · *first walk speed* = *second walk total time* · *second walk speed*

The text at the bottom of page 124 and the top of page 125 in your book expresses the relationship you discovered as a multiplication equation and as two proportions.

(continued)

You can use what you know about solving proportions to show that the three equations are equivalent. For example, to show that the first proportion is equivalent to the multiplication equation, multiply both sides of the proportion first by *second walk total time* and then by *second walk speed.* (Try it!)

In the speed versus time investigation, the product of the speed and total time was constant. Read Example A in your book carefully. It discusses another relationship in which two variables have a constant product. Such a relationship is called an **inverse variation,** and the variables are said to be *inversely proportional.*

The equation for an inverse variation can be written in the form $xy = k$ or $y = \frac{k}{x}$, where x and y are the inversely proportional variables and k is the constant product, called the **constant of variation.** The graph of an inverse variation is always curved and never crosses the x- or y-axis.

EXAMPLE

If the area of a triangle remains constant, the length of the base is inversely proportional to the height. For a given area, if a triangle has height 8 cm then the base is 4.5 cm. If the height is 25 cm, then the base is 1.44 cm.

a. What is the base of a triangle with height 6 cm, if the area remains constant?

b. What is the height of a triangle with base 30 cm, if the area remains constant?

▶ ***Solution***

Let h represent the height, and let b represent the length of the base. Because h and b are inversely proportional, they have a constant product k. Because $h = 8$ when $b = 4.5$, k must be $8 \cdot 4.5$ or 36. You can now write the inverse variation equation as $b = \frac{36}{h}$.

a. To find the base of a triangle with height 6, use the equation and substitute 6 for h.

$b = \frac{36}{h}$ Original equation.

$b = \frac{36}{6}$ Substitute 6 for h.

$b = 6$ Divide.

So the base is 6 cm.

b. To find the height if the base is 30, substitute 30 for b.

$30 = \frac{36}{h}$ Original equation.

$\frac{1}{30} = \frac{h}{36}$ To get h in the numerator, invert both ratios.

$36 \cdot \frac{1}{30} = h$ Multiply by 36 to undo the division.

$1.2 = h$ Multiply and divide.

The height of the triangle is 1.2 cm.

CONDENSED LESSON

2.7 Evaluating Expressions

In this lesson you will

- apply the **order of operations** to evaluate expressions
- use **algebraic expressions** to explain number tricks

The **order of operations** specifies the order in which the operations in an expression should be evaluated. For example, to evaluate $12 - 2 \cdot 5$, you multiply first and then subtract, so the result is 2. In your text, read the rules for Order of Operations on page 135.

You can write expressions that apply a sequence of operations to a starting number. Here is an example.

EXAMPLE Write a mathematical expression, using x as the starting number, that represents this sequence of operations: Multiply 12 by a starting number; then subtract the answer from 16; divide this result by 4; and then subtract that answer from 50.

▶ ***Solution*** You can organize your work in a table.

Description	Expression
Starting value.	x
Multiply by 12.	$12x$
Subtract the answer from 16.	$16 - 12x$
Divide this result by 4.	$\frac{16 - 12x}{4}$
Subtract the answer from 50.	$50 - \frac{16 - 12x}{4}$

The fraction bar is a grouping symbol meaning that the entire numerator is divided by 4.

An expression that involves both numbers and variables is called an **algebraic expression.** You'll work with algebraic expressions in the investigation.

Investigation: Number Tricks

Read the introduction to the investigation on page 136. Pick a number and try the trick yourself, using your calculator. Try several times, picking different numbers each time. What do you notice? The result is the same every time!

Steps 1–4 Number tricks like the one in the introduction to the investigation work because certain operations, such as multiplication and division, get "undone" in the course of the trick. Here is the algebraic expression for this trick.

$$\frac{3(x + 9) - 6}{3} - x$$

Steps 5–7 Here is the algebraic expression representing the sequence shown in Step 5 in the book.

$$\frac{2(n + 3) - 4}{2} - n + 2$$

(continued)

The result of this trick will always be the original number, because the given steps undo each other.

Step 8 Invent your own number trick with at least five stages. Use your calculator to test your number trick with a variety of starting numbers. When you are convinced that it works, try it on a few people.

In this lesson you saw two kinds of number tricks: those in which the result is always a given number, and those in which the final number is the same as the number you started with. Both types of number trick work because operations are "undone." In the case of the number tricks where the final result is a given number, often the number that you start with is subtracted away at some point in the process.

Read Example B in your book and make sure that you understand it. Then read Example C.

In Example C, you see that although addition and subtraction happen at the same time in the order of operations, you must be careful about how you evaluate. The expression $7 - 4 + 2$ is not the same as $7 - (4 + 2)$. (This is because subtracting $(4 + 2)$ is like subtracting both the 4 and the 2.) Although you can add and subtract in any order, it is often useful to think of subtraction as adding a negative. If you think of the expression as $7 + (-4) + 2$, you'll be more likely to evaluate it correctly. This strategy can be especially useful when describing the steps in a complicated number trick. Consider the expression $\frac{5 - 3(x + 2)}{3} + x$. Think of this instead as $\frac{5 + -3(x + 2)}{3} + x$. Now you can write the steps in this way.

Pick a number.

Add 2.

Multiply by -3.

Add 5.

Divide by 3.

Add the original number.

CONDENSED LESSON

2.8 Undoing Operations

In this lesson you will

- work backward to **solve equations**

There is another kind of number trick that is a little different from the ones you learned about in the preceding lesson. In this type of trick, no matter what the ending number is, you can figure out the starting number. For example, pick a number and do this sequence of operations: Add 3, multiply by 7, subtract 4, divide by 2, and add 1. If your final result is 13, you must have started with 1. If your final result is 0, you must have started with -2. You can use the process of **undoing operations** to solve tricks like this.

Investigation: Just Undo It!

Steps 1–4 Consider this description: "I took my secret number, added 6, divided by 2, subtracted 9, and multiplied by 5. The result was -10." Can you figure out the starting number? To do so, fill in a chart like the one below. Notice how parentheses are used to indicate the proper grouping.

Description	Sequence	Expression		
Picked a number.	?	x		
Added 6.	Ans + 6	$x + 6$		
Divided by 2.	Ans / 2	$\frac{x+6}{2}$		
Subtracted 9.	Ans − 9	$\frac{x+6}{2} - 9$		
Multiplied by 5.	Ans · 5	$5\left(\frac{x+6}{2} - 9\right)$		

Remember which operations undo each other: Subtraction and addition undo each other, and division and multiplication undo each other. Fill in another column to indicate the operation that undoes each operation in the Sequence column. Then work up the chart, starting with the result of -10, and undo each operation until you find the starting number. In this case the starting number was 8.

Description	Sequence	Expression	Undo	Result
Picked a number.	?	x		8
Added 6.	Ans + 6	$x + 6$	$-\ (6)$	14
Divided by 2.	Ans / 2	$\frac{x+6}{2}$	$\cdot\ (2)$	7
Subtracted 9.	Ans − 9	$\frac{x+6}{2} - 9$	$+\ (9)$	-2
Multiplied by 5.	Ans · 5	$5\left(\frac{x+6}{2} - 9\right)$	/ (5)	-10

(continued)

Steps 5–6 An **equation** is a statement that says the value of one expression is equal to the value of another expression. You can use an undo chart to solve the equation $7 + \frac{x-3}{4} = 42$.

Equation: $7 + \frac{x-3}{4} = 42$		
Description	**Undo**	**Result**
Pick x.		143
$-$ (3)	$+$ (3)	140
/ (4)	$\cdot$ (4)	35
$+$ (7)	$-$ (7)	42

In this investigation you learned about one method of **solving equations.** Make sure you understand the difference between an equation and an expression. The value of a variable that makes an equation true is a **solution** to the equation. For some number trick equations, *every* number is a solution. However, this is not usually the case. For example, 4 is the only solution to $2x + 3 = 11$.

Now work through Example B in your book. It shows how to represent a real-world situation with an equation, and then solve the equation by working backward, undoing each operation until you reach the solution.

